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On subordination for classes of non-Bazilevič type

Abstract. We give some subordination results for new classes of normalized analytic functions containing differential operator of non-Bazilevič type in the open unit disk. By using Jack's lemma, sufficient conditions for this type of operator are also discussed.

1. Introduction and preliminaries. Consider the functions F in the open disk $U := \{z \in \mathbb{C} : |z| < 1\}$, defined by

$$\begin{aligned} F(z) &= \frac{z^\alpha}{(1-z)^\alpha} = \sum_{n=0}^{\infty} \frac{(\alpha)_n}{n!} z^{n+\alpha} \\ &= z^\alpha + \sum_{n=1}^{\infty} \frac{(\alpha)_n}{n!} z^{n+\alpha} \\ &= z^\alpha + \sum_{n=2}^{\infty} \frac{(\alpha)_{n-1}}{(n-1)!} z^{n+\alpha-1}, \qquad \alpha \geq 1. \end{aligned} \tag{1.1}$$

From (1.1), assuming α to be a parameter with the values $\alpha := \frac{n+m}{m}$, $m \in \mathbb{N}$, and having $n = 0$ in the first term of the series, we can write F in the form

$$F(z) = z + \sum_{n=2}^{\infty} \frac{(\alpha)_{n-1}}{(n-1)!} z^{n+\alpha-1}. \tag{1.2}$$

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By employing (1.2), we define classes of analytic functions of fractional power.

Let $\mathcal{A}_\alpha^+$ be the class of all normalized analytic functions F in the open disk U of the form

$$F(z) = z + \sum_{n=2}^{\infty} a_{n,\alpha} z^{n+\alpha-1}, \quad \alpha \geq 1,$$

satisfying $F(0) = 0$ and $F'(0) = 1$. Moreover, let $\mathcal{A}_\alpha^-$ be the class of all normalized analytic functions F in the open disk U of the form

$$F(z) = z - \sum_{n=2}^{\infty} a_{n,\alpha} z^{n+\alpha-1}, \quad a_{n,\alpha} \geq 0; \quad n = 2, 3, \dots,$$

satisfying $F(0) = 0$ and $F'(0) = 1$.

Definition 1.1 (Subordination Principle). For two functions f and g analytic in U, we say that the function f is subordinate to g in U and write $f(z) \prec g(z)$ $(z \in U)$, if there exists a Schwarz function $w(z)$ analytic in U with $w(0) = 0$, and $|w(z)| < 1$, such that $f(z) = g(w(z))$, $z \in U$. In particular, if the function g is univalent in U, the above subordination is equivalent to $f(0) = g(0)$ and $f(U) \subset g(U)$.

Now we define a differential operator as follows:

$$\begin{aligned} D_\alpha^0 F(z) &= F(z) = z + \sum_{n=2}^{\infty} a_{n,\alpha} z^{n+\alpha-1}, \ \ \alpha \geq 1, \\ D_\alpha^1 F(z) &= \frac{F(z)}{2} + \frac{zF'(z)}{2} = z + \sum_{n=2}^{\infty} \frac{(n+\alpha)}{2} a_{n,\alpha} z^{n+\alpha-1}, \\ &\vdots \\ D_\alpha^k F(z) &= D\big(D^{k-1}F(z)\big) = z + \sum_{n=2}^{\infty} \left[\frac{(n+\alpha)}{2}\right]^k a_{n,\alpha} z^{n+\alpha-1}. \end{aligned} \tag{1.3}$$

Let $\mathcal{A}$ be the class of analytic functions of the form $f(z) = z + a_2 z^2 + \dots$. Obradovič [8] introduced a class of functions $f \in \mathcal{A}$ such that for $0 < \mu < 1$,

$$\Re\left\{ f'(z) \left(\frac{z}{f(z)}\right)^{\mu} \right\} > 0, \quad z \in U. \tag{1.4}$$

He called it the class of function of non-Bazilevič type. There are many subordination results for this class (see [15]). In fact, this type of functions has been used to solve various problems (see [14]).

The main object of the present work is to apply a method based on the differential subordination in order to derive sufficient conditions for functions $F \in \mathcal{A}_\alpha^+$ and $F \in \mathcal{A}_\alpha^-$ to satisfy

$$(1.5) \qquad \left(D_\alpha^k F(z)\right)' \left(\frac{z}{D_\alpha^k F(z)}\right)^\mu \prec q(z), \quad D_\alpha^k F(z) \neq 0, \quad z \in U,$$

where q is a given univalent function in U such that $q(z) \neq 0$, $\mu \neq 0$.

Moreover, we give applications of these results in fractional calculus. We shall need the following known results:

Lemma 1.1 ([4]). *Let $q(z)$ be univalent in the unit disk U and θ and ϕ be analytic in a domain D containing $q(U)$ with $\phi(w) \neq 0$ when $w \in q(U)$. Set $Q(z) := zq'(z)\phi(q(z))$, $h(z) := \theta(q(z)) + Q(z)$. Suppose that*
1. *$Q(z)$ is starlike univalent in U, and*
2. *$\Re \frac{zh'(z)}{Q(z)} > 0$ for $z \in U$.*

If $\theta(p(z)) + zp'(z)\phi(p(z)) \prec \theta(q(z)) + zq'(z)\phi(q(z))$, then $p(z) \prec q(z)$ and q is the best dominant.

Lemma 1.2 ([5]). *Let $q(z)$ be convex univalent in the unit disk U and ψ and $\gamma \in \mathbb{C}$ with $\Re\{1 + \frac{zq''(z)}{q'(z)} + \frac{\psi}{\gamma}\} > 0$. If $p(z)$ is analytic in U and $\psi p(z) + \gamma z p'(z) \prec \psi q(z) + \gamma z q'(z)$, then $p(z) \prec q(z)$ and q is the best dominant.*

2. Subordination results. In this section, we study subordination for normalized analytic functions in the classes $\mathcal{A}_\alpha^+$ and $\mathcal{A}_\alpha^-$.

Theorem 2.1. *Let a function q be univalent in the unit disk U such that $q(z) \neq 0$, $\frac{zq'(z)}{q(z)}$ is starlike univalent in U and*

$$(2.1) \quad \Re\left\{1 + \frac{zq''(z)}{q'(z)} - \frac{zq'(z)}{q(z)} + \frac{a}{bq(z)}\right\} > 0, \ b \neq 0, \ q'(z) \neq 0, \quad z \in U.$$

If $F \in \mathcal{A}_\alpha^+$ satisfies the subordination

$$\frac{a}{\left(D_\alpha^k F(z)\right)'}\left(\frac{D_\alpha^k F(z)}{z}\right)^\mu + b\left[\mu(1 - \frac{z(D_\alpha^k F(z))'}{D_\alpha^k F(z)}) + \frac{z(D_\alpha^k F(z))''}{(D_\alpha^k F(z))'}\right] \prec \frac{a}{q(z)} + b\frac{zq'(z)}{q(z)},$$

then

$$\left(D_\alpha^k F(z)\right)' \left(\frac{z}{D_\alpha^k F(z)}\right)^\mu \prec q(z)$$

and q is the best dominant.

Proof. Let the function p be defined by

$$p(z) := \left(D_\alpha^k F(z)\right)' \left(\frac{z}{D_\alpha^k F(z)}\right)^\mu, \quad D_\alpha^k F(z) \neq 0, \quad z \in U.$$

By setting

$$\theta(\omega) := \frac{a}{\omega} \text{ and } \phi(\omega) := \frac{b}{\omega}, \quad b \neq 0,$$

it can easily be observed that $\theta(\omega)$ is analytic in $\mathbb{C} - \{0\}$, $\phi(\omega)$ is analytic in $\mathbb{C} - \{0\}$ and that $\phi(\omega) \neq 0$, $\omega \in \mathbb{C} - \{0\}$. Also we obtain

$$Q(z) = zq'(z)\phi(q(z)) = \frac{bzq'(z)}{q(z)}$$

and

$$h(z) = \theta(q(z)) + Q(z) = \frac{a}{q(z)} + b\frac{zq'(z)}{q(z)}.$$

It is clear that $Q(z)$ is starlike univalent in U,

$$\Re\left\{\frac{zh'(z)}{Q(z)}\right\} = \Re\left\{1 + \frac{zq''(z)}{q'(z)} - \frac{zq'(z)}{q(z)} + \frac{a}{bq(z)}\right\} > 0.$$

By straightforward computation, we have

$$\begin{aligned} \frac{a}{p(z)} + b\frac{zp'(z)}{p(z)} &= \frac{a}{\big(D_\alpha^k F(z)\big)'}\left(\frac{D_\alpha^k F(z)}{z}\right)^\mu \\ &\quad + b\left[\mu\left(1 - \frac{z\big(D_\alpha^k F(z)\big)'}{D_\alpha^k F(z)}\right) + \frac{z\big(D_\alpha^k F(z)\big)''}{\big(D_\alpha^k F(z)\big)'}\right] \\ &\prec \frac{a}{q(z)} + b\frac{zq'(z)}{q(z)}. \end{aligned}$$

Then by the assumption of the theorem, we see that the assertion of the theorem follows by application of Lemma 1.1. □

Corollary 2.1. *Assume that* (2.1) *holds and q is convex univalent in U. If $F \in \mathcal{A}_\alpha^+$ and*

$$\begin{aligned} \frac{a}{\big(D_\alpha^k F(z)\big)'}\left(\frac{D_\alpha^k F(z)}{z}\right)^\mu + b\left[\mu\left(1 - \frac{z\big(D_\alpha^k F(z)\big)'}{D_\alpha^k F(z)}\right) + \frac{z(D_\alpha^k F(z))''}{(D_\alpha^k F(z))'}\right] \\ \prec a\left(\frac{1+Bz}{1+Az}\right)^\mu + b\frac{\mu z(A-B)}{(1+Az)(1+Bz)}, \end{aligned}$$

then

$$\big(D_\alpha^k F(z)\big)'\left(\frac{z}{D_\alpha^k F(z)}\right)^\mu \prec \left(\frac{1+Az}{1+Bz}\right)^\mu, \quad -1 \le B < A \le 1$$

and $q(z) = \left(\frac{1+Az}{1+Bz}\right)^\mu$ *is the best dominant.*

Corollary 2.2. *Assume that* (2.1) *holds and q is convex univalent in U. If $F \in \mathcal{A}_\alpha^+$ and*

$$\frac{a}{\big(D_\alpha^k F(z)\big)'}\left(\frac{D_\alpha^k F(z)}{z}\right)^\mu + b\left[\mu\left(1-\frac{z\big(D_\alpha^k F(z)\big)'}{D_\alpha^k F(z)}\right)+\frac{z\big(D_\alpha^k F(z)\big)''}{(D_\alpha^k F(z))'}\right] \prec a\left(\frac{1-z}{1+z}\right)^\mu + \frac{2\mu b z}{1-z^2},$$

for $z \in U$, $\mu \neq 0$, then

$$\big(D_\alpha^k F(z)\big)'\left(\frac{z}{D_\alpha^k F(z)}\right)^\mu \prec \left(\frac{1+z}{1-z}\right)^\mu$$

and $q(z) = \left(\frac{1+z}{1-z}\right)^\mu$ is the best dominant.

Corollary 2.3. *Assume that* (2.1) *holds and q is convex univalent in U. If $F \in \mathcal{A}_\alpha^+$ and*

$$\frac{a}{\big(D_\alpha^k F(z)\big)'}\left(\frac{D_\alpha^k F(z)}{z}\right)^\mu + b\left[\mu\left(1-\frac{z\big(D_\alpha^k F(z)\big)'}{D_\alpha^k F(z)}\right)+\frac{z\big(D_\alpha^k F(z)\big)''}{\big(D_\alpha^k F(z)\big)'}\right] \prec a e^{-\mu A z} + \mu b A z$$

for $z \in U$, $\mu \neq 0$, then

$$\big(D_\alpha^k F(z)\big)'\left(\frac{z}{D_\alpha^k F(z)}\right)^\mu \prec e^{\mu A z}$$

and $q(z) = e^{\mu A z}$ is the best dominant.

The next result can be found in [3].

Corollary 2.4. *Assume that $k = 0$ in Theorem 2.1. Then*

$$(F(z))'\left(\frac{z}{F(z)}\right)^\mu \prec q(z)$$

and q is the best dominant.

Theorem 2.2. *Let a function $q(z)$ be convex univalent in the unit disk U such that $q'(z) \neq 0$ and*

$$\Re\left\{1+\frac{zq''(z)}{q'(z)}+\frac{1}{\gamma}\right\} > 0, \quad \gamma \neq 0. \tag{2.2}$$

Suppose that $\big(D_\alpha^k F(z)\big)'\left(\frac{z}{D_\alpha^k F(z)}\right)^\mu$ is analytic in U. If $F \in \mathcal{A}_\alpha^-$ satisfies the subordination

$$\big(D_\alpha^k F(z)\big)'\left(\frac{z}{D_\alpha^k F(z)}\right)^\mu\left[\mu\gamma\left(1-\frac{z\big(D_\alpha^k F(z)\big)'}{D_\alpha^k F(z)}\right)+\frac{z\big(D_\alpha^k F(z)\big)''}{\big(D_\alpha^k F(z)\big)'}\right] \prec q(z)+\gamma z q'(z),$$

then

$$\big(D_\alpha^k F(z)\big)'\left(\frac{z}{D_\alpha^k F(z)}\right)^{\mu} \prec q(z), \quad z \in U, \quad D_\alpha^k F(z) \neq 0$$

and q is the best dominant.

Proof. Let the function p be defined by

$$p(z) := \left(\frac{z}{D_\alpha^k F(z)}\right)^{\mu}, \quad D_\alpha^k F(z) \neq 0, \quad z \in U.$$

By setting $\psi = 1$, it can easily be observed that

$$\begin{aligned} &p(z) + \gamma z p'(z) \\ &= \big(D_\alpha^k F(z)\big)'\left(\frac{z}{D_\alpha^k F(z)}\right)^{\mu}\left[\mu\gamma\left(1 - \frac{z\big(D_\alpha^k F(z)\big)'}{D_\alpha^k F(z)}\right) + \frac{z\big(D_\alpha^k F(z)\big)''(z)}{\big(D_\alpha^k F(z)\big)'}\right] \\ &\prec q(z) + \gamma z q'(z). \end{aligned}$$

Then by the assumption of the theorem we see that the assertion of the theorem follows by application of Lemma 1.2. □

Corollary 2.5. *Assume that* (2.2) *holds and q is convex univalent in U. If $F \in \mathcal{A}_\alpha^-$ and*

$$\begin{aligned} \big(D_\alpha^k F(z)\big)'\left(\frac{z}{D_\alpha^k F(z)}\right)^{\mu}&\left[\mu\gamma\left(1 - \frac{z\big(D_\alpha^k F(z)\big)'}{D_\alpha^k F(z)}\right) + \frac{z\big(D_\alpha^k F(z)\big)''(z)}{\big(D_\alpha^k F(z)\big)'}\right] \\ &\prec \left(\frac{1+Az}{1+Bz}\right)^{\mu} + \mu\gamma z(A-B)\frac{(1+Az)^{\mu-1}}{(1+Bz)^{\mu+1}}, \end{aligned}$$

then

$$\big(D_\alpha^k F(z)\big)'\left(\frac{z}{D_\alpha^k F(z)}\right)^{\mu} \prec \left(\frac{1+Az}{1+Bz}\right)^{\mu}, \quad -1 \le B < A \le 1$$

and $q(z) = \left(\frac{1+Az}{1+Bz}\right)^{\mu}$ is the best dominant.

Corollary 2.6. *Assume that* (2.2) *holds and q is convex univalent in U. If $F \in \mathcal{A}_\alpha^-$ and*

$$\begin{aligned} \big(D_\alpha^k F(z)\big)'\left(\frac{z}{D_\alpha^k F(z)}\right)^{\mu}&\left[\mu\gamma\left(1 - \frac{z\big(D_\alpha^k F(z)\big)'}{D_\alpha^k F(z)}\right) + \frac{z\big(D_\alpha^k F(z)\big)''(z)}{\big(D_\alpha^k F(z)\big)'}\right] \\ &\prec \left[\frac{1+z}{1-z}\right]^{\mu}\left\{1 + \frac{2\gamma\mu z}{1-z^2}\right\} \end{aligned}$$

for $z \in U$, $\mu \neq 0$, then

$$\big(D_\alpha^k F(z)\big)'\left(\frac{z}{D_\alpha^k F(z)}\right)^{\mu} \prec \left(\frac{1+z}{1-z}\right)^{\mu}$$

and $q(z) = \left(\frac{1+z}{1-z}\right)^{\mu}$ is the best dominant.

Corollary 2.7. *Assume that* (2.2) *holds and* q *is convex univalent in* U. *If* $F \in \mathcal{A}_\alpha^-$ *and*

$$\left(D_\alpha^k F(z)\right)' \left(\frac{z}{D_\alpha^k F(z)}\right)^\mu \left[\mu\gamma\left(1 - \frac{z\left(D_\alpha^k F(z)\right)'}{D_\alpha^k F(z)}\right) + \frac{z\left(D_\alpha^k F(z)\right)''(z)}{(D_\alpha^k F(z))'}\right] \prec e^{\mu A z}(1 + \mu\gamma A z)$$

for $z \in U$, $\mu \neq 0$, *then*

$$\left(D_\alpha^k F(z)\right)' \left(\frac{z}{D_\alpha^k F(z)}\right)^\mu \prec e^{\mu A z}$$

and $q(z) = e^{\mu A z}$ *is the best dominant.*

The next result can be found in [3].

Corollary 2.8. *Assume that* $k = 0$ *in Theorem 2.2. Then*

$$(F(z))' \left(\frac{z}{F(z)}\right)^\mu \prec q(z)$$

and q *is the best dominant.*

3. Applications. In this section, we present some applications of Section 2 to fractional integral operators. Assume that $f(z) = \sum_{n=2}^{\infty} \varphi_n z^{n-1}$ and let us begin with the following definitions:

Definition 3.1 ([12]). The fractional integral of order α is defined, for a function f, by

$$I_z^\alpha f(z) := \frac{1}{\Gamma(\alpha)} \int_0^z f(\zeta)(z-\zeta)^{\alpha-1} d\zeta, \quad \alpha \geq 1,$$

where the function f is analytic in a simply-connected region of the complex z-plane ($\mathbb{C}$) containing the origin and the multiplicity of $(z-\zeta)^{\alpha-1}$ is removed by requiring $\log(z-\zeta)$ to be real when $(z-\zeta) > 0$.

Note that (see [12], [7])

$$I_z^\alpha z^\mu = \frac{\Gamma(\mu+1)}{\Gamma(\mu+\alpha+1)} z^{\mu+\alpha}, \quad (\mu > -1).$$

Thus we have

$$I_z^\alpha f(z) = \sum_{n=2}^{\infty} a_n z^{n+\alpha-1}$$

where $a_n := \frac{\varphi_n \Gamma(n)}{\Gamma(n+\alpha)}$, for all $n = 2, 3, \ldots$. This implies that $z + I_z^\alpha f(z) \in \mathcal{A}_\alpha^+$ and $z - I_z^\alpha f(z) \in \mathcal{A}_\alpha^-$ ($\varphi_n \geq 0$), so we get the following results:

Theorem 3.1. *Let the assumptions of Theorem 2.1 be satisfied. Then*

$$D_\alpha^k\big(z+I_z^\alpha f(z)\big)'\left(\frac{z}{D_\alpha^k(z+I_z^\alpha f(z))}\right)^\mu \prec q(z), \quad z\neq 0, \quad z\in U$$

and q is the best dominant.

Proof. Consider the function F be defined by

$$F(z) := z + I_z^\alpha f(z), \quad z\in U,\ z\neq 0. \qquad \square$$

Theorem 3.2. *Let $k=0$ in Theorem 2.2. Then*

$$D_\alpha^k\big(z-I_z^\alpha f(z)\big)'\left(\frac{z}{D_\alpha^k(z-I_z^\alpha f(z))}\right)^\mu \prec q(z), \quad z\neq 0,\ z\in U$$

and q is the best dominant.

Proof. Consider the function F be defined by

$$F(z) := z - I_z^\alpha f(z), \quad z\in U,\ z\neq 0. \qquad \square$$

Let $F(a,b;c;z)$ be the Gauss hypergeometric function (see [13]) defined, for $z\in U$, by

$$F(a,b;c;z)=\sum_{n=0}^{\infty}\frac{(a)_n(b)_n}{(c)_n(1)_n}z^n,$$

where is the Pochhammer symbol defined by

$$(a)_n := \frac{\Gamma(a+n)}{\Gamma(a)} = \begin{cases} 1, & (n=0); \\ a(a+1)(a+2)\dots(a+n-1), & (n\in\mathbb{N}). \end{cases}$$

We need the following definition of fractional operators of the Saigo type fractional calculus (see [10], [9]).

Definition 3.2. For $\alpha>0$ and $\beta,\eta\in\mathbb{R}$, the fractional integral operator $I_{0,z}^{\alpha,\beta,\eta}$ is defined by

$$I_{0,z}^{\alpha,\beta,\eta}f(z)=\frac{z^{-\alpha-\beta}}{\Gamma(\alpha)}\int_0^z (z-\zeta)^{\alpha-1}F\left(\alpha+\beta,-\eta;\alpha;1-\frac{\zeta}{z}\right)f(\zeta)d\zeta$$

where the function $f(z)$ is analytic in a simply-connected region of the z-plane containing the origin, with the order

$$f(z)=O(|z|^\epsilon)(z\to 0), \quad \epsilon>\max\{0,\beta-\eta\}-1$$

and the multiplicity of $(z-\zeta)^{\alpha-1}$ is removed by requiring $\log(z-\zeta)$ to be real when $z-\zeta>0$.

From Definition 3.2, with $\beta < 0$, we have

$$\begin{aligned}
I_{0,z}^{\alpha,\beta,\eta} f(z) &= \frac{z^{-\alpha-\beta}}{\Gamma(\alpha)} \int_0^z (z-\zeta)^{\alpha-1} F\left(\alpha+\beta, -\eta; \alpha; 1-\frac{\zeta}{z}\right) f(\zeta) d\zeta \\
&= \sum_{n=0}^{\infty} \frac{(\alpha+\beta)_n (-\eta)_n}{(\alpha)_n (1)_n} \frac{z^{-\alpha-\beta}}{\Gamma(\alpha)} \int_0^z (z-\zeta)^{\alpha-1} \left(1-\frac{\zeta}{z}\right)^n f(\zeta) d\zeta \\
&:= \sum_{n=0}^{\infty} B_n \frac{z^{-\alpha-\beta-n}}{\Gamma(\alpha)} \int_0^z (z-\zeta)^{n+\alpha-1} f(\zeta) d\zeta \\
&= \sum_{n=0}^{\infty} B_n \frac{z^{-\beta-1}}{\Gamma(\alpha)} f(\zeta) \\
&:= \frac{\overline{B}}{\Gamma(\alpha)} \sum_{n=2}^{\infty} \varphi_n z^{n-\beta-1}
\end{aligned}$$

where $\overline{B} := \sum_{n=0}^{\infty} B_n$. Denote $a_n := \frac{\overline{B}\varphi_n}{\Gamma(\alpha)}$, $\forall\, n = 2, 3, \ldots$, and let $\alpha = -\beta$. Thus $z + I_{0,z}^{\alpha,\beta,\eta} f(z) \in \mathcal{A}_\alpha^+$ and $z - I_{0,z}^{\alpha,\beta,\eta} f(z) \in \mathcal{A}_\alpha^-$ ($\varphi_n \geq 0$), so we have the following results:

Theorem 3.3. *Assume that the hypotheses of Theorem 2.1 are satisfied. Then*

$$D_\alpha^k \big(z + I_{0,z}^{\alpha,\beta,\eta} f(z)\big)' \left(\frac{z}{D_\alpha^k (z + I_{0,z}^{\alpha,\beta,\eta} f(z))}\right)^\mu \prec q(z), \quad z \neq 0,\ z \in U$$

and q is the best dominant.

Proof. Consider the function F defined by

$$F(z) := z + I_{0,z}^{\alpha,\beta,\eta} f(z), \quad z \in U,\ z \neq 0. \qquad \square$$

Theorem 3.4. *Assume that the hypotheses of Theorem 2.2 are satisfied. Then*

$$D_\alpha^k \big(z - I_{0,z}^{\alpha,\beta,\eta} f(z)\big)' \left(\frac{z}{D_\alpha^k (z - I_{0,z}^{\alpha,\beta,\eta} f(z))}\right)^\mu \prec q(z), \quad z \neq 0,\ z \in U$$

and q is the best dominant.

Proof. Consider the function F defined by

$$F(z) := z - I_{0,z}^{\alpha,\beta,\eta} f(z), \quad z \in U,\ z \neq 0. \qquad \square$$

Remark 3.1. Note that the authors have recently studied and defined several other classes of analytic functions related to fractional power (see [2], [1], [4]).

4. The class $\mathcal{S}_\mu(\gamma)$. A function $F(z) \in \mathcal{A}_\alpha^+$ is said to be in the class $\mathcal{S}_\mu(\gamma)$ if it satisfies

$$\big(D_\alpha^k F(z)\big)' \left(\frac{z}{D_\alpha^k F(z)}\right)^\mu \prec \frac{1+z}{1-\gamma z}, \quad (z \in U,\ \gamma \neq 1).$$

To discuss our problem, we have to recall here the following lemma due to Jack [15].

Lemma 4.1. *Let w be analytic in U with $w(0) = 0$. If $|w(z)|$ attains its maximum value on the circle $|z| = r < 1$ at a point z_0, then*

$$z_0 w'(z_0) = k w(z_0),$$

where k is a real number and $k \geq 1$.

We get the following result:

Theorem 4.1. *If $F \in \mathcal{A}_\alpha^+$ satisfies*

$$\Re\left[\mu - \mu\frac{z\big(D_\alpha^k F(z)\big)'}{D_\alpha^k F(z)} + \frac{z\big(D_\alpha^k F(z)\big)''}{\big(D_\alpha^k F(z)\big)'}\right] < \frac{1+\gamma}{2(1-\gamma)}, \quad (z \in U) \tag{4.1}$$

for some $0 < \gamma < 1$, $0 < \mu < 1$, then $F(z) \in \mathcal{S}_\mu(\gamma)$.

Proof. Let w be defined by

$$\big(D_\alpha^k F(z)\big)' \left(\frac{z}{D_\alpha^k F(z)}\right)^\mu = \frac{1+w(z)}{1-\gamma w(z)}, \quad (1 \neq \gamma w(z)).$$

Then $w(z)$ is analytic in U with $w(0) = 0$. It follows that

$$\begin{aligned}\Re\left[\mu - \mu\frac{z\big(D_\alpha^k F(z)\big)'}{D_\alpha^k F(z)} + \frac{z\big(D_\alpha^k F(z)\big)''}{D_\alpha^k F(z))'}\right] &= \Re\left[\frac{z(\gamma w'(z)+1)}{(1-\gamma w(z))(1+w(z))}\right] \\ &< \frac{1+\gamma}{2(1-\gamma)}, \quad \gamma \neq 1.\end{aligned}$$

Now we proceed to prove that $|w(z)| < 1$. Suppose that there exists a point $z_0 \in U$ such that

$$\max_{|z| \leq |z_0|} |w(z)| = |w(z_0)| = 1.$$

Then, using Lemma 4.1 and letting $w(z_0) = e^{i\theta}$ and $z_0 w'(z_0) = k e^{i\theta}$, $k \geq 1$, we obtain

$$\begin{aligned}\Re\left[\mu - \mu\frac{z\big(D_\alpha^k F(z_0)\big)'}{D_\alpha^k F(z_0)} + \frac{z_0\big(D_\alpha^k F(z_0)\big)''}{D_\alpha^k F(z_0))'}\right] &= \Re\left[\frac{z_0(w'(z_0)\gamma+1)}{(1-\gamma w(z_0))(1+w(z_0))}\right] \\ &= \Re\left[\frac{k e^{i\theta}\gamma+1}{(1-\gamma e^{i\theta})(1+e^{i\theta})}\right] \\ &= \frac{k(\gamma+1)}{2(1-\gamma)} \geq \frac{1+\gamma}{2(1-\gamma)},\end{aligned}$$

$0 < \gamma < 1$. Thus we have

$$\Re\left[\mu - \mu\frac{z\big(D_\alpha^k F(z)\big)'}{D_\alpha^k F(z)} + \frac{z\big(D_\alpha^k F(z)\big)''}{D_\alpha^k F(z))'}\right] \geq \frac{1+\gamma}{2(1-\gamma)}, \quad (z \in U)$$

which contradicts the hypothesis (4.1). Therefore, we conclude that $|w(z)| < 1$ for all $z \in U$ that is

$$\big(D_\alpha^k F(z)\big)'\left(\frac{z}{D_\alpha^k F(z)}\right)^\mu \prec \frac{1+z}{1-\gamma z}, \quad (z \in U,\ \gamma \neq 1).$$

This completes the proof of the theorem. □

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References

[1] Darus, M., Ibrahim, R. W., *Coefficient inequalities for a new class of univalent functions*, Lobachevskii J. Math. **29**(4) (2008), 221–229.

[2] Ibrahim, R. W., Darus, M., *On subordination theorems for new classes of normalize analytic functions*, Appl. Math. Sci. (Ruse) **2**(56) (2008), 2785–2794.

[3] Ibrahim, R. W., Darus, M., *Subordination for new classes of non-Bazilevič type*, UNRI-UKM Symposium, KE-4 (2008).

[4] Ibrahim, R. W., Darus, M., *Differential subordination results for new classes of the family $\mathcal{E}(\Phi, \Psi)$*, JIPAM. J. Ineq. Pure Appl. Math. **10**(1) (2009), Art. 8, 9 pp.

[5] Jack, I. S., *Functions starlike and convex of order k*, J. London Math. Soc. **3** (1971), 469–474 .

[6] Miller, S. S., Mocanu, P. T., *Differential Subordinantions. Theory and Applications*, Monographs and Textbooks in Pure and Applied Mathematics, 225, Marcel Dekker, Inc., New York, 2000.

[7] Miller, K. S., Ross, B., *An Introduction to the Fractional Calculus and Fractional Differential Equations*, John Wiley and Sons, Inc., New York, 1993.

[8] Obradovič, M., *A class of univalent functions*, Hokkaido Math. J. **27**(2) (1998), 329–335.

[9] Raina, R. K., *On certain class of analytic functions and applications to fractional calculus operator*, Integral Transform. Spec. Funct. **5** (1997), 247–260.

[10] Raina, R. K., Srivastava, H. M., *A certain subclass of analytic functions associated with operators of fractional calculus*, Comput. Math. Appl. **32** (1996), 13–19.

[11] Shanmugam, T. N., Ravichangran, V. and Sivasubramanian, S., *Differential sandwich theorems for some subclasses of analytic functions*, Austral. J. Math. Anal. Appl. **3**(1) (2006), 1–11.

[12] Srivastava, H. M., Owa, S. (Eds.), *Univalent Functions, Fractional Calculus, and Their Applications*, Halsted Press, John Wiley and Sons, New York, Chichester, Brisbane, Toronto, 1989.

[13] Srivastava, H. M., Owa, S. (Eds.), *Current Topics in Analytic Function Theory*, World Scientific Publishing Company, Singapore, New Jersey, London, Hong Kong, 1992.

[14] Tuneski, N., Darus, M., *Fekete-Szegö functional for non-Bazilevič functions*, Acta Math. Acad. Paedagog. Nyházi. (N.S.) **18**(2) (2002), 63–65.

[15] Wang, Z., Gao, C. and Liao, M., *On certain generalized class of non-Bazilevič functions*, Acta Math. Acad. Paedagog. Nyházi. (N.S.) **21**(2) (2005), 147–154.

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